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Pointwise Convergence of Wavelet Expansions*

GILBERT G. WALTER[†]

Department of Mathematical Sciences, University of Wisconsin–Milwaukee, Box 413, Milwaukee, Wisconsin 53201

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The expansion of a distribution or function in regular orthogonal wavelets is considered. The expansion of a function is shown to converge uniformly on compact subsets of intervals of continuity. The expansion of a distribution is shown to converge pointwise to the value of the distribution where it exists. © 1995 Academic Press, Inc.

1. INTRODUCTION

Orthogonal wavelets have only been around for a few years but have already proved very useful in attacking both applied and theoretical problems. Their usefulness in applications is largely based on their efficient representation of signals; fewer coefficients are needed than with other methods. This arises in turn from their localization properties in both time and frequency. These are well known results and have been extensively studied [C, D, M1, M2].

In the theoretical domain, orthogonal wavelets have proved useful as unconditional bases for certain Banach spaces [D]. Part of this arises because of the convergence properties of the wavelet expansions which are superior to those of expansions in classical orthogonal series.

In this work we shall consider these convergence properties as they relate to pointwise convergence. We consider expansions of both functions and tempered distributions and show that the partial sums converge pointwise under very general conditions.

We shall assume the standard conditions on the wavelets hold, namely that there is a scaling function $\varphi(t)$ whose translates $\{\varphi(t-n)\}$ are orthogonal. We shall also assume that $\varphi(t)$ is r-regular [M2], i.e., $\varphi \in C^r(\mathbb{R})$ and

$$|\varphi^{(k)}(t)| \leqslant \frac{C_{kp}}{(1+|t|)^p}, \qquad k=0, 1, ..., r; \quad p=0, 1, ...$$

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The space of r-regular functions with a suitable topology is usualy denoted S_r. Its dual space S'_r consists of tempered distributions of order $\leqslant r$ [Z2]. Associated with φ is a multiresolution analysis $\{V_m\}$ of subspaces of $L^2(\mathbb{R})$. The mother wavelet $\psi(t)$ based on $\varphi(t)$ gives rise to the orthonormal basis $\{\psi_{mn}\}$ of $L^2(\mathbb{R})$, where

$$\psi_{mn}(t) = 2^{m/2}\psi(2^m t - n), \qquad m, n \in \mathbb{Z}. \tag{1}$$

Each $f \in L^2(\mathbb{R})$ has two representations,

$$f(t) = \sum_m \sum_n b_{mn}\psi_{mn}(t) \tag{2}$$

and

$$f(t) = \sum_n a_{mn} 2^{m/2}\varphi(2^m t - n) + \sum_{k=m}^{\infty} \sum_n b_{kn}\psi_{kn}(t) = f_m(t) + r_m(t), \tag{3}$$

where convergence is in the sense of $L^2(\mathbb{R})$. The function $f_m \in V_m$ and in fact is the projection of f onto V_m. It is also given in terms of the reproducing kernel $q_m(x, t)$ of V_m as

$$f_m(x) = \int_{-\infty}^{\infty} q_m(x, t) f(t)\, dt. \tag{4}$$

The function q_m is given by

$$q_m(x, t) = 2^m q(2^m x, 2^m t)$$

where

$$q(x, t) = \sum_n \varphi(x - n)\, \varphi(t - n).$$

We shall be primarily concerned with the convergence of f_m to f. This is studied by considering the reproducing kernel sequence $\{q_m(x, y)\}$ which is a delta sequence. It was shown in [W1] that this converges to $\delta(x - y)$ in the Sobolev space $H^{-\alpha}$ for $\alpha > \frac{1}{2}$ at a rate of $O(2^{-m(\alpha - 1/2)})$ under quite general conditions. Here we show that this and certain other sequences are quasi-positive delta sequences.

These delta sequences have a number of properties not shared by those arising from other orthogonal systems. These are to some extent a consequence of the following properties of $q(x, y)$ [M2, p. 33]:

$$\text{(i)} \quad q(x + 1, y + 1) = q(x, y)$$

$$\text{(ii)} \quad \left| \frac{\partial^{\alpha + \beta}}{\partial x^\alpha\, \partial y^\beta} q(x, y) \right| \leqslant C_m(1 + |x - y|)^{-m}, \quad 0 \leqslant \alpha, \beta \leqslant r, \ m \in \mathbb{N}, \tag{5}$$

$$\text{(iii)} \quad \int_{-\infty}^{\infty} q(x, y)\, y^\alpha\, dy = x^\alpha, \qquad 0 \leqslant \alpha \leqslant r.$$

This last condition is perhaps the most surprising; it enables us to deduce that the expansion in $\{\varphi(t-n)\}$ of any polynomial of degree $\leqslant r$ is equal to it, while for other functions it is only an approximation. In particular, we have

$$1=\sum_n \varphi(x-n)\int \varphi(y-n)\,dy=\sum_n \varphi(x-n)\int \varphi(y)\,dy \tag{6}$$

and hence

$$\sum_n \varphi(x-n)=C, \qquad x\in\mathbb{R},$$

where C is a constant $\neq 0$. From this we get

$$\int_0^1 C\,dx=\int_0^1 \sum_n \varphi(x-n)\,dx=\int_{-\infty}^{\infty} \varphi(x)\,dx,$$

which when substituted into (6) gives $1=C^2$. Thus we may, by changing the sign if necessary, conclude that

$$1=\int_{-\infty}^{\infty} \varphi(x)\,dx=\sum_n \varphi(x-n).$$

2. Quasi-positive Delta Sequences

A *quasi-positive delta sequence* is a sequence $\{\delta_m(\cdot, y)\}$ of functions in $L^1(\mathbb{R})$ with parameter $y\in\mathbb{R}$ which satisfy the following:

(i) there is a $C>0$ such that

$$\int_{-\infty}^{\infty} |\delta_m(x, y)|\,dx\leqslant C, \qquad y\in\mathbb{R}, \quad m\in\mathbb{N};$$

(ii) there is a $c>0$ such that

$$\int_{y-c}^{y+c} \delta_m(x, y)\,dx\to 1 \tag{7}$$

uniformly on compact subsets of $\mathbb{R}$, as $m\to\infty$;

(iii) for each $\gamma > 0$,

$$\sup_{|x-y| \geq \gamma} |\delta_m(x, y)| \to 0 \quad \text{as} \quad m \to \infty.$$

It is easy to check that $\delta_m(x, y) \to \delta(x-y)$ in the sense of tempered distributions for fixed y.

An example of a quasi-positive delta sequence is the Fejer kernel,

$$F_m(x, y) = \left(\sin^2 \frac{m+1}{2}(x-y)\right) \Big/ \left(2(m+1)\,\pi \sin^2 \frac{x-y}{2}\right) \chi_{[-\pi, \pi]}(x-y),$$

while a delta sequence that is *not* quasi-positive is the Dirichlet kernel of Fourier series,

$$D_m(x, y) = \frac{\sin(m+1/2)(x-y)}{2\pi \sin((x-y)/2)} \chi_{[-\pi, \pi]}(x-y).$$

LEMMA 1. *Let $\{\delta_m(x, y)\}$ be a quasi-positive delta sequence and let $f \in L^1(\mathbb{R})$ be continuous on (a, b); then*

$$f_m(y) = \int_{-\infty}^{\infty} \delta_m(x, y) f(x)\, dx \to f(y) \qquad \textit{as} \quad m \to \infty$$

uniformly on compact subsets of (a, b).

The proof of this lemma is similar to that for positive delta sequences of the form $\delta_m(x, y) = \delta_m(x-y)$ given for bounded intervals in [Z1, p. 87]. We present it here in the interest of completeness.

Let $\gamma > 0$; then

$$\begin{aligned} f_m(y) &= \int_{y-\gamma}^{y+\gamma} \delta_m(x, y) f(x)\, dx + \int_{y+\gamma}^{\infty} + \int_{-\infty}^{y-\gamma} \\ &= f(y) \int_{y-\gamma}^{y+\gamma} \delta_m(x, y)\, dx \\ &\quad + \int_{y-\gamma}^{y+\gamma} \delta_m(x, y)(f(x) - f(y))\, dx + \left\{ \int_{y+\gamma}^{\infty} + \int_{-\infty}^{y-\gamma} \right\} \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{8}$$

Now let K be a compact subset and let $[\alpha, \beta] \subset (a, b)$ be a closed subinterval containing K, and let $y \in [\alpha, \beta]$. We choose γ in (8) so that $0 < \gamma < c$,

$\beta+\gamma<b$, and $\alpha-\gamma>a$, where c is the constant in (ii) of (7). Then for any $0<\varepsilon<1$, we restrict γ further so that

$$|f(x)-f(y)|<\varepsilon \quad \text{for} \quad y\in[\alpha,\beta] \quad \text{and} \quad |(x-y)|<\gamma.$$

From this we deduce that

$$|I_2| \leqslant \varepsilon \int_{y-\gamma}^{y+\gamma} |\delta_m(x,y)|\,dx \tag{9}$$

and that

$$|I_3| \leqslant \sup_{\gamma\leqslant|x-y|} |\delta_m(x,y)|\,\|f\|_1 < \varepsilon\,\|f\|_1, \qquad m\geqslant M_1, \tag{10}$$

where we have chosen M_1 so large that $\sup_{\gamma\leqslant|x-y|}|\delta_m(x,y)|<\varepsilon$ for $m\geqslant M_1$. We then choose $M_2\geqslant M_1$ so large that

$$\left|1-\int_{y-\gamma}^{y+\gamma}\delta_m(x,y)\,dx\right|<\varepsilon, \qquad m\geqslant M_2. \tag{11}$$

This is possible since by (7)(iii)

$$\int_{y+\gamma}^{y+c}\delta_m(x,y)\,dx\to 0$$

uniformly on $\mathbb{R}$ as $m\to\infty$. We now combine (9), (10), and (11) to get

$$\begin{aligned}|f(y)-f_m(y)| &\leqslant |f(y)-I_1|+|I_2|+|I_3| \\ &\leqslant |f(y)|\left|1-\int_{y-\gamma}^{y+\gamma}\delta_m(x,y)\,dx\right| \\ &\quad+\varepsilon\int_{-\infty}^{\infty}|\delta_m(x,y)|\,dx+\varepsilon\,\|f\|_1 \\ &\leqslant \sup_{y\in[\alpha,\beta]}|f(y)|\,\varepsilon+\varepsilon C+\varepsilon\,\|f\|_1\end{aligned}$$

for $m\geqslant M_2$, which gives us the desired uniform convergence on $[\alpha,\beta]$ and hence on K.

Remark 1. The same proof gives us pointwise convergence at a single point x_0 provided only that f is continuous at that point.

We now show that the reproducing kernels $q_m(x,y)$ constitute a quasi-positive delta sequence.

LEMMA 2. *Let $q_m(x, y)$ be the reproducing kernel of V_m, $\varphi \in S_r$; then $\{q_m(x, y)\}$ satisfies* (7)(i), (ii), *and* (iii).

The inequality (7)(i) follows from the fact that

$$\begin{aligned}\int_{-\infty}^{\infty} |q_m(x, y)|\, dx &= \int_{-\infty}^{\infty} 2^m\, |q(2^m x, 2^m y)|\, dx \\ &= \int_{-\infty}^{\infty} |q(x, 2^m y)|\, dx \\ &\leqslant C_2 \int_{-\infty}^{\infty} (1 + |x - 2^m y|)^{-2}\, dx = C\end{aligned}$$

by (5)(ii). The second condition (ii) is obtained by a change of scale; let $c > 0$, then

$$\begin{aligned}\int_{y-c}^{y+c} q_m(x, y)\, dx &= \int_{2^m(y-c)}^{2^m(y+c)} q(x, 2^m y)\, dx \\ &= \int_{t-2^m c}^{t+2^m c} q(x, t)\, dx \\ &= 1 - \int_{t+2^m c}^{\infty} - \int_{-\infty}^{t-2^m c} = 1 - I_1 - I_2.\end{aligned}$$

The integral I_1, by (5)(ii) again, satisfies

$$|I_1| \leqslant C_2 \int_{t+2^m c}^{\infty} \frac{1}{1 + (t-x)^2}\, dx = C_2 \int_{2^m c}^{\infty} \frac{1}{1 + x^2}\, dx \to 0$$

as $m \to \infty$, and the same is true for I_2. The third condition also follows from the same inequality for any $\gamma > 0$.

COROLLARY 1. *Let $f \in L^1 \cap L^2(\mathbb{R})$ be continuous on (a, b); let f_m be the projection of f onto V_m; then*

$$f_m \to f \quad \text{as} \quad m \to \infty$$

uniformly on compact subsets of (a, b).

This follows from the two lemmas since the projection is given by

$$f_m(y) = \int_{-\infty}^{\infty} q_m(x, y)\, f(x)\, dx.$$

Remark 2. This result is valid under much more general hypotheses. In fact, we have not used the r-regularity of $\varphi(t)$ (but will in the next section). Nor for that matter have we used the dilation property of $\varphi(t)$. In fact, the only properties we need are those of

$$\text{(i)} \quad \sum_n \varphi(t-n) = 1,$$

$$\text{(ii)} \quad |\varphi(t)| \leqslant C_{03}(1+|t|)^{-3}.$$

An anonymous referee has pointed out that such general results have long been known in the area of spline approximation [BJ, CJW]. The difference here is in the emphasis and in the method of proof based on quasi-positive delta sequences.

Remark 3. The results in this section are for a single dimension. However, the proofs are valid in higher dimensions for certain types of $\varphi(\mathbf{t})$. In particular, if $\varphi(\mathbf{t}) = \varphi_1(t_1) \cdots \varphi_d(t_d)$, and the integrals are appropriately reinterpreted, then the results in Corollary 1 hold.

3. Local Convergence of Distribution Expansions

The global convergence of the expansions of tempered distributions (in the sense of S') is important for theoretical purposes. However, for computational purposes, it is desirable to have some sort of local convergence. In the case of distributions we should like to have the expansions converge to the value of the distribution at a point [L]. This is a weaker version of the continuity of a function at a point.

Definition 1. Let $f \in S'$; f is said to have a value γ of order r at x_0 if there exists a continuous function $F(x)$ of polynomial growth such that its distribution derivative $D^r F = f$ in some neighborhood of x_0 and

$$\lim_{x \to x_0} \frac{F(x)}{(x-x_0)^r} = \frac{\gamma}{r!}.$$

Examples. The δ distribution has value 0 everywhere except at 0; the function $\sin(1/x)$ considered as an element of S' has value 0 at 0.

Lemma 3. *Let $q_m(x, t)$ be the reproducing kernel of V_m, with $\varphi \in S_r$, and let $\alpha \in \mathbb{Z}$, $0 \leqslant \alpha \leqslant r$; then*

$$K_m(x, t) = \frac{(x-t)^\alpha}{\alpha!} \frac{\partial^\alpha}{\partial t^\alpha} q_m(x, t)$$

is a quasi-positive delta sequence on $\mathbb{R}$.

Proof. We must show that the three conditions given in (7) hold. To prove (i) of (7) we observe first that since $|\varphi^{(\alpha)}(x)| \leqslant C_{\alpha k}(1+|x|)^{-k}$,

$$\int |K_0(x,t)|\,dt \leqslant \sum_{k=0}^{\alpha} \binom{\alpha}{k} \sum_n |\varphi(x-n)|\,|x-n|^k \int |\varphi^{(\alpha)}(t)| \frac{|t|^{\alpha-k}}{\alpha!}\,dt \leqslant C,$$

a constant. The same holds for $K_m(x,t)$ since

$$\begin{aligned} K_m(x,t) &= \frac{[2^m(x-t)]^{\alpha}}{\alpha!} \frac{\partial^{\alpha}}{\partial(2^m t)^{\alpha}} 2^m q_0(2^m x, 2^m t) \\ &= 2^m K_0(2^m x, 2^m t). \end{aligned}$$

To prove (ii) of (7) we observe first that

$$\int_{-\infty}^{\infty} K_m(x,y)\,dy = \int_{-\infty}^{\infty} q_m(x,y)\,dy = 1$$

by integration by parts and (6). Thus it remains to be shown that, for some $c>0$,

$$\left\{ \int_{x+c}^{\infty} + \int_{-\infty}^{x-c} \right\} K_m(x,y)\,dy \to 0$$

as $m \to \infty$ uniformly for x in bounded set. This is shown by the same sorts of tricks as in the proof of (i):

$$\begin{aligned} \int_{x+c}^{\infty} K_m(x,y)\,dy &= \int_{2^m(x+c)}^{\infty} K_0(2^m x, z)\,dz \\ &= \sum_n \sum_{k=0}^{\alpha} (2^m x - n)^k \binom{\alpha}{k} \varphi(2^m x - n) \\ &\quad \times \int_{2^m(x+c)}^{\infty} \frac{\varphi^{(\alpha)}(z-n)}{\alpha!} (n-z)^{\alpha-k}\,dz. \end{aligned} \tag{12}$$

This last integral satisfies

$$\begin{aligned} \left| \int_{2^m(x+c)}^{\infty} \varphi^{(\alpha)}(z-n)(z-n)^{\alpha-k}\,dz \right| &\leqslant \int_{2^m(x+c)-n}^{\infty} \frac{C_{\alpha_j}}{(1+|z|)^j} |z|^{\alpha-k}\,dz \\ &\leqslant \frac{C_{\alpha_j}}{(1+|2^m(x+c)-n|)^{j-\alpha+k-2}} \int_{-\infty}^{\infty} \frac{1}{(1+|z|)^2}\,dz, \qquad j \geqslant \alpha-k+2. \end{aligned}$$

Hence (12) is bounded by

$$\left|\int_{x+c}^{\infty} K_m(x, y)\, dy\right| \leqslant C_p \sum_n \frac{1}{(|2^m x - n| + 1)^p} \frac{1}{(1 + |2^m(x+c) - n|)^p}$$

for all $p \geqslant 1$. Since

$$\frac{1}{1+|x-a|} \frac{1}{1+|x-b|} \leqslant \frac{1}{1+|a-b|},$$

it follows that for $p > 2$

$$\begin{aligned}\left|\int_{x+c}^{\infty} K_m(x, y)\, dy\right| &\leqslant \frac{C_p}{1+2^m c} \sum_n \frac{1}{(1+|2^m x - n|)^{p-1}} \frac{1}{(1+|2^m(x+c)-n|)^{p-1}} \\ &= \frac{C_p}{1+2^m c} h_p(2^m x, 2^m(x+c)),\end{aligned}$$

where

$$h_p(x, y) = \sum \frac{1}{(1+|x-n|)^{p-1}} \frac{1}{(1+|y-n|)^{p-1}}$$

is uniformly bounded in x and y. Hence (ii) holds. For (iii) of (7) we need a bound

$$\begin{aligned}\sup_{|x-y| \geqslant \gamma} |K_m(x, y)| &\leqslant \sup_{|x-y| \geqslant \gamma} 2^m \sum_n |\varphi(2^m x - n) \frac{(x-y)^\alpha}{\alpha!} 2^{\alpha m} \varphi^{(\alpha)}(2^m y - n)| \\ &= \sup_{|z-w| \geqslant 2^m \gamma} 2^m \sum_n |\varphi(z-n)| \frac{|z-w|^\alpha}{\alpha!} |\varphi^{(\alpha)}(w-n)| \\ &\leqslant \sup_{|z-w| \geqslant 2^m \gamma} 2^m \frac{C}{(1+|z-w|)^2} h_p(z, w) \leqslant \frac{C\, \|h_p\|_\infty\, 2^m}{(1+2^m \gamma)^2}\end{aligned}$$

given by the same argument as that for (ii).

THEOREM 1. *Let $f \in S'_{r-1}$ and have a value γ of order $\alpha \leqslant r$ at $x = x_0$. Then the function f_m given by* (4) *satisfies*

$$f_m(x_0) \to \gamma \quad as \quad m \to \infty.$$

Proof. Each $f \in S'_{r-1}$ is a derivative of order $\beta \leqslant r$ of a continuous function G of polynomial growth. We may assume, by adding a polynomial of degree $<\beta$ if necessary and by increasing α if necessary, that $G = F$ and $\alpha = \beta$. Then

$$\begin{aligned} f_m(x) &= \int_{-\infty}^{\infty} (-1)^\alpha \, \partial_y^\alpha q_m(x, y) \, F(y) \, dy \\ &= \int_{-\infty}^{\infty} \frac{(x-y)^\alpha}{\alpha!} \partial_y^\alpha q_m(x, y) \frac{F(y) \alpha!}{(y-x)^\alpha} dy \\ &= \int_{x-A}^{x+A} + \int_{x+A}^{\infty} + \int_{-\infty}^{x-A}. \end{aligned}$$

By a repetition of the argument in (12) we see that

$$\int_{x+A}^{\infty} + \int_{-\infty}^{x-A} \to 0 \qquad \text{as} \quad m \to \infty.$$

Hence we may express f_m as

$$f_m(x) = \int_{-\infty}^{\infty} K_m(x, y) \, F_A(x, y) \, dy + o(1)$$

where $F_A(x, y)$ is continuous for all y except for $y = x \pm A$ and has compact support. Clearly F_A is bounded as well. Hence

$$f_m(x_0) \to F_A(x_0, x_0) \qquad \text{as} \quad m \to \infty$$

by the property of quasi-positive delta sequences given in Remark 1. Since $F_A(x_0, x_0) = \gamma$, the theorem is proved.

Concluding Remark. We have only considered one type of partial sum of the wavelet expansion which can be expressed as

$$f_m(t) = \sum_{k=-\infty}^{m-1} \sum_{n=-\infty}^{\infty} b_{kn} \psi_{kn}(t).$$

We could also truncate the inner series; this presents no problem since $\psi_{mn}(t) = 2^{m/2} \psi(2^m t - n)$ and $\psi(t)$ is rapidly decreasing as $|t| \to \infty$. Hence the partial sums of the form

$$S_{m, n_1, n_2}(t) = \sum_{k=-\infty}^{m} \sum_{n=-n_1}^{n_2} b_{kn} \psi_{kn}(t)$$

converge whenever the hypotheses of Corollary 1 or Theorem 1 are satisfied.

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